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An interactive learning activity for the formation of the concept of function based on representational transfer

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Abstract

The concept of function is a central idea in mathematics. Functions have many facets, which often cause problems to students. On the one hand the nature of functional dependencies has various aspects: the mapping aspect, the aspect of change, and the object aspect. On the other hand there are various representations for functions. Each representation emphasizes different aspects of the functional dependency. To establish subconcepts like injectivity within the function concept some representations are more suitable than others. One needs to integrate these subconcepts in the concept of function by transferring them to other representations.

We present an interactive learning environment for the conceptualization of the notions of "function", "injectivity", "surjectivity", and "bijection", using a three-stage approach basing on different representations and linking them dynamically.

The learning environment "Squiggle-M" allows the integration of the mentioned subconcepts in the concept of function. "Squiggle-M" is a mathematical exploration tool offering a bundle of experimentation laboratories. Different representation forms of functions are implemented using an interactive geometry software. The environment also presents a collection of open study questions that can be answered within the laboratories by making use of different representation forms. The individual learning process of the student is reflected by the software's feedback module based on an intelligent (semi-)automated assessment system.

1. Motivation and Rationale

The concept of function is essential in mathematics and mathematics education. It usually causes many problems for different reasons: The concept is very complex containing many facets, aspects, subconcepts, representations etc. For example Vollrath [17] and similarly Dubinsky & Harel [6] describe the following aspects of functional dependencies:

- *The mapping aspect:* functions seen as point wise relations, static view of functional dependencies
- *The aspect of change:* dynamic view of functional dependencies in sense of 'What effect does the change of a value have on the change of another value?'
- *The object aspect:* functions seen as a whole with global characteristics and functions seen as algebraic objects

Furthermore there are several representations for functions, e.g. words, tables, formula, arrow or ladder diagrams, graphs etc. Each representation relates to one or more aspect in an eminent way. For example tables or ladder diagrams emphasize the mapping aspect whereas graphs relate to the dynamic or object aspect. A teacher has to decide which representation is suitable within a certain context. E.g., when talking about ‘monotonicity’ a graph is suitable whereas a (static) ladder diagram is not. For the conceptualization of the notion of function representational transfer is essential. As mentioned by many authors (e.g. Swan [15], Janvier [11], Kerslake [13]) representational transfer is one of the main problems of students.

The large variety of representations often lead to the development of isolated - sometimes even contradictory - mental images of the notion of function. For example Dreyfus & Vinner [4] describe that students develop a *concept image* of the notion function resulting from interaction with examples and non-examples. The *concept image* is the set of all mental images and characteristic properties in connection with the term of function. The *concept image* often contains images like: a function needs to be one rule, or a function has to look reasonable, i.e. monotonous and without jump discontinuities. This often contradicts the *concept definition* of function, which is the definition one would give for the notion of function. Dreyfus & Vinner [4] showed that students use their *concept image* to decide if a given example is a function or not. Even if students give a correct definition for the term of function they do not use it as a basis for their decision about examples and non-examples. A typical problem is shown in Figure 1: At the beginning of a study, which is partly described in Hoffkamp [10], 10th grade secondary school students of age 15 to 16 were asked to write a letter to an imaginary friend where they describe the term of function. Figure 1 shows an excerpt of such a letter.

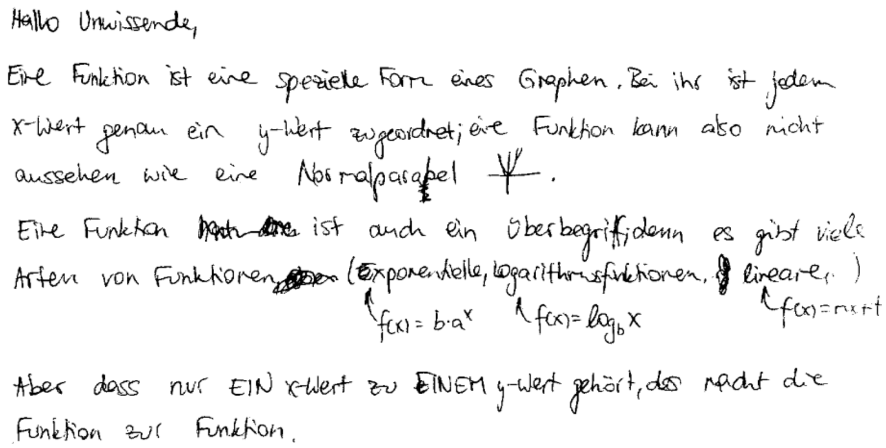
Hallo Unwissende,
Eine Funktion ist eine spezielle Form eines Graphen. Bei ihr ist jedem x-Wert genau ein y-Wert zugeordnet; eine Funktion kann also nicht aussehen wie eine Normalparabel.
Eine Funktion ist auch ein Überbegriff, denn es gibt viele Arten von Funktionen (Exponentielle, logarithmusfunktionen, lineare,)
$f(x) = b \cdot a^x$ $f(x) = \log_b x$ $f(x) = mx+t$
Aber dass nur EIN x-Wert zu EINEM y-Wert gehört, das macht die Funktion zur Funktion.

Figure 1: *Dear ignoramous, a function is a special form of a graph where each x value has exactly one assigned y value; therefore a function cannot look like a parabola. [...] But that only ONE x value belongs to ONE y value makes the function a function. (student, grade 10, secondary school)*

Although the student gives a correct definition for the term of function the subconcepts injectivity/bijectivity and uniqueness are confused. Why do students have problems with the definition? The concept definition of a function is usually introduced in grade 8 (age 13)– based on Dirichlet’s definition – as follows:

A function is an assignment rule by which each element x of a set is assigned to exactly one element y of (another) set.

This definition is very abstract and general. In grade 8 students usually work on problems like ‘Is the following a function or not?’ and it is not obvious for the students why functions ‘need’ to be defined like that. The above definition is the result of a long development in mathematics following the demand of universality and precision (Fischer & Malle [8]) and the genesis of that notion is usually concealed from the students. E.g. Euler’s definition of function contained the subconcept of continuity by stating that functions must be ‘drawable by hand in one move’. The need of universality led to the omission of this property. Also the aspect of change is not obvious in the

Dirichlet definition. While certain aspects and subconcepts are neglected due to generality and precision, they need to be (re-)formulated and integrated in the concept of function.

2. Aims and main ideas

The aim of the presented work is the introduction and integration of the subconcepts *uniqueness, totality, injectivitiy, surjectivity* and *bijectivity* by using interactive learning activities for visualization and experimentation.

The introduction of these subconcepts is usually done by using finite arrow diagrams, which seems to be the most suitable representation form for this purpose. But of course these diagrams do not represent ‘common’ real functions. To integrate the above subconcepts in the concept of function one has to recover them in other representation forms and connect them to other subconcepts. For example the subconcept of monotonicity is strongly connected to the subconcept injectivity as monotonous functions are injective. But finite arrow diagrams are neither suitable for depicting continuous functions nor for representing monotonicity. Finite arrow diagrams can be extended to ladder diagrams [9]. This implies in a sense the introduction of continuity although one still depicts a finite number of assignments. By dynamizing ladder diagrams the subconcepts injectivity and monotonicity can be explored locally. But only by using graphs their global connection is best seen. Therefore one core idea of the learning environment is the visualization of the connection between arrow diagrams and graphs.

2.1 The role of visualizations in mathematics and mathematics education

The idea of interactive visualization is central for this work. Visualizations play an important role in mathematics or like stated in Fischer & Malle [8]:

To some extend mathematics takes place in an interaction between representation, interpretation and operation.

Therefore the physical representation or visualization of abstractions is an essential property of mathematics or as Kaput [12] says:

The fundamental premise is that the root phenomena of mathematics learning and application are concerned with representation and symbolization because these are at the heart of the content of mathematics and are simultaneously at the heart of the cognitions associated with mathematical activity. (p. 22)

Mathematicians often do not carry out abstractions only in their mind but search for visual representations like symbols on a sheet of paper or diagrams etc. Visualization allows extended elaboration by focussing on certain aspects and abstracting from other aspects. Furthermore they facilitate visual communication. While language is linear, complex issues need to be brought in a succession to be described and explained verbally. This often neglects some important dependencies of the subject matter. In contrast visualizations allow the presentation of complex issues as a whole side by side. For example different aspects of functions can be explored and observed together in their relationship to each other. We use the epistemological and heuristic aspects of visualizations and representations in a special designed learning environment.

2.2 Automated and semi automated assessment

In common learning environments the teacher is not able to support and analyze the whole learning process of each student. Especially at the university level the number of students is very large. Hence, the teacher cannot give individual feedback to each of them. As a consequence, assessment is usually done classically by weekly homework and central tests. Often the students just get the results of their work without any deeper comment. A personalized feedback given by a tutor or the teacher can only occur in individual meetings. But the students need to know about their individual solution process – successful or faulty – just in time.

Analyzing mistakes and learning how to avoid or resolve them should be as well integral part of the learning content as developing correct solution strategies. This is particularly essential for students whose aim is to become teachers at school.
To enable an individual analysis of a learning process in lectures with many students, new concepts for teaching are necessary. In the project SAiL-M (Semi-automated Analysis of individual Learning processes in Mathematics, founded by the German Federal Ministry for Education and Research, BMBF) learning environments and adapted software tools for student oriented and activating lessons in mathematics are developed (Bescherer et al. [1]). First prototypes of computer-aided learning tools with integrated intelligent automated and semi-automated assessment are implemented for different mathematical topics like algebra of sets, congruencies and line reflections, basic geometric proofs, or proofs based on complete induction. The tools give automated feedback on standard solutions and standard mistakes and detect non-standard answers given by the students. Hence the teacher is disburdened from the correction of standard solutions. She or he can concentrate on unusual strategies and errors that cannot be detected by the automated assessment system or that need a deeper discussion in the lecture. The tools are used in beginner courses for teacher students at the university level.

3. The tool concept and methodological strategy

The formation and understanding of new concepts is optimally driven by an approach that allows the students to discover properties, to connect those properties to logical relations, and that encourage the students to work with these properties and concepts (cf. Vollrath [16]). To deal with all three aspects, we developed a tool concept that includes the following dimensions:
Different representations serve as a basis for the formation of the concept of function. Relations can be pointed out and connected to mental images of already known properties. Furthermore, practising the transition between different representations supports the development of functional thinking.
Working experimentally is conducive to the intuitional understanding of concepts. By adapting and working on concrete self-constructed examples depicting a special situation, the student gets more insights to different properties or concepts. Properties and concepts can be developed and defined by different demonstrating examples. Different types of learners can be supported individually in their formation and portability of understanding.
Open questions offer space for individual learning. The student can use the tool on his own preferred way.
Intelligent assessment and self-determination. Intelligent assessment provides feedback that accompanies the students' learning process.. According to the ideas of feedback-on-demand (Bescherer & Spannagel [3]), the student decides on her or his own whether she or he needs feedback or not. We regard the student as being self-determined learner. Being self-determined is an important motivational factor. This feedback assists the self-reflection of the learner. It supports the teacher since she or he can concentrate on unusual solutions and mistakes. In our setting a solution or a mistake is called unusual when, first, it is rarely produced by students, and, second, the software has no implemented recognition algorithm to analyze it. Whenever the software cannot give feedback adequately, the system notifies the student and asks him or her to contact the teacher. On the other hand the student is informed about his own strategies and mistakes during her or his learning process. He or she is advised to deal with sources and correcting of errors. Hence, the student advances his or her mathematical competencies on the one hand. And on another level he or she develops didactical knowledge about preventing and dealing with mistakes at the same time.
Adaptivity enables the teacher to adjust the tool easily to the needs of his students and his lecture or course. The available sequence of exercises can be selected and sorted. Special notions can be chosen according to the used textbook if necessary. Adding new questions and the generation of own exercises should be possible.

Methodologically we use a *three-stage approach* for the conceptualization of the notions *uniqueness, totality, injectivity, surjectivity* and *bijectivity*.

- *Stage one:* The above subconcepts are introduced via finite arrow diagrams by focusing on the mapping aspect of functions.
- *Stage two:* Extended ladder diagrams are used as continuous analogues to finite arrow diagrams. This allows the integration of the aspects of continuity and change by using dynamic visualizations.
- *Stage three:* Extended ladder diagrams are dynamically linked to common graphs as a third representation. The above subconcepts can be explored in both representations by using the dynamic link. This leads to the integration of the object aspect of the notion of functions because global properties like monotonicity can be recognized best in the global view of the function graph.

This *three stage approach* is realised in the learning environment *Squiggle-M* as follows.

4. Technical realisation – The learning environment *Squiggle-M*

Squiggle-M is an open learning environment for the formation of the concept of function. It was developed for teacher students in the first year at university level in Baden-Württemberg, Germany. *Squiggle-M* is a *Java* application implemented with the *Yacht-M* framework [7]. The software offers collection of open learning laboratories. Each laboratory consists of one question, problem, or challenge and one or more embedded interactive diagrams depicting different representations of functions. Those diagrams are based on the dynamic geometry software *Cinderella 2.1* [14].
We distinguish between two different types of laboratories. *Assignment laboratories* are based on finite assignments, which can be defined interactively by the user. Properties of the assignments can be discovered. *Representation laboratories* prepare the transfer between assignment diagrams and function graphs.
The formation of concepts is supported by experimentation questions adapted to some of the laboratories. The questions are part of the learning environment, but can be selected and adapted by the teacher according to his course.
Each laboratory includes an adapted automated assessment. Additionally, the student can ask for individual feedback from his or her teacher or tutor via e-mail whenever the automated feedback does not answer a question the student has. An appropriate e-mail functionality is integrated into the software, containing the possibility to add a screenshot of the actually examined situation.

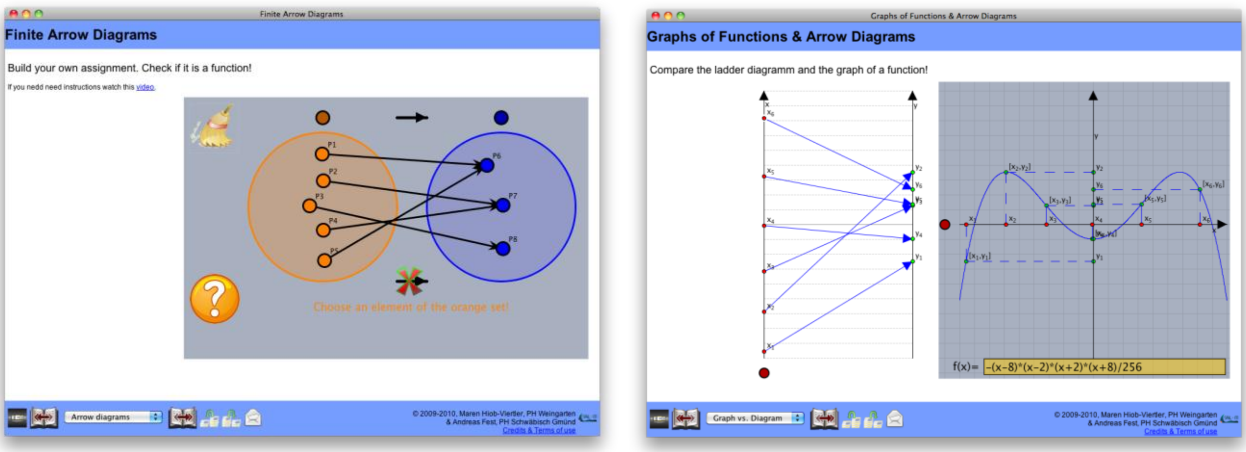

Figure 2: *Squiggle-M.* a) Assignment laboratory, b) Representation laboratory

4.1 The assignment laboratory

The assignment laboratory (Figure 2.a) offers the possibility to create different assignments interactively. On different self-created examples the concept of function can be explored. Orange and blue points can be dragged into the pre-image and image sets. The orange points can be assigned to the blue ones by connecting them with arrows. By using the question mark symbol the properties of the defined assignment are proved automatically. The user gets feedback whether the assignment defines a function or not. The feedback engine generates informative messages like "*This is not a function because the assignment disregards the uniqueness/totality*." or "*This is a injective/surjective/bijective function*."

The assignment laboratory can be used for further investigations by working on different exploration exercises. The student gets a statement like "*If the size of the pre-image set A greater than the size of the image set B then there exists an injective function from A on B*" (see Figure 3). She or he has to decide whether this claim is true or false. Therefore, the student has to construct several examples or counterexamples using the interactive assignment diagram. Using the camera button she or he can log in the actual example. The feedback engine checks the student's solution in three stages:

1. The validity of all examples respectively counterexamples is tested when they are logged-in by the student. The next checks are executed by clicking the question mark icon.
2. Is the number of entered examples sufficient for a successful learning process?
3. Is the final answer correct?

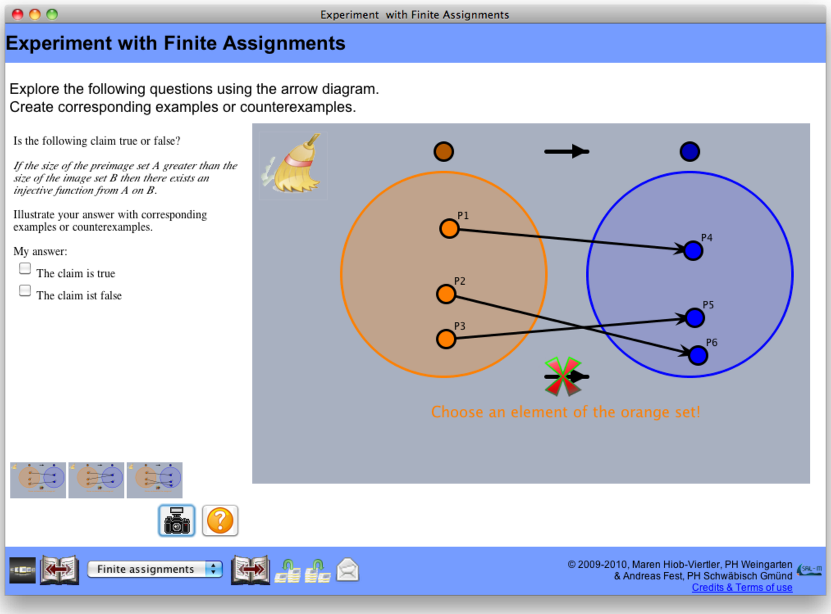

Figure 3: *Squiggle-M*. Assignment laboratory with exploration question

Finally, an assessment laboratory offers the possibility to test her or his knowledge about the definition of functions. The test laboratory generates random assignments. The user must decide whether or not the given assignment is a function and which properties it has. She or he must reason her or his answer by using the definitions. In the case that the assignment is no function the user is asked to manipulate the interactive diagram in order to produce an example for a function.

4.2 The representation laboratory

The connection between ordinary assignment diagrams and well-known function graphs (and their corresponding function equations) is a central aspect of the representation laboratory (Figure 2.b).

The user enters a function equation and displays either the function graph in a coordinate system or a dynamic ladder diagram [9] depicting the assignment defined by the function.
The user can drag several points onto the x-axis of both representations. The corresponding y-value is shown on the y-axis and the point (x,y) is marked on the function graph. All x-values can be dragged interactively on the x-axis of both representations while the y-values are updated automatically.
In the representation laboratory the two representations ladder diagram and function graph can either be displayed simultaneously side by side, or just one type of representation can be chosen. In the latter case, an animated change of representation is possible, pointing out the connection of types, which is important for the concept formation.
The actual version of Squiggle-M uses the representation laboratory just as a visualization tool for graphs. To transfer the knowledge in the concepts of injectivity, surjectivity and bijectivity to continuous functions, applicable exercises with adapted feedback based on the representation laboratory will be developed in a next step.

5. Outlook

The tool *Squiggle-M* is used at the Universities of Education Weingarten, Ludwigsburg, and Heidelberg for first year arithmetic courses. The software is employed for demonstrating and depicting functional properties during the lecture as well as for attending student exercises.
In our educational setting (see Bescherer, Spannagel & Müller [4]) new concepts are first shortly introduced in a lecture. Afterwards, the students acquire and deepen these concepts once again by working alone or in small groups on a bundle of possible exercises. The students choose on their own when, where and which exercises they solve. Some assisted tutorial sessions are offered to the students where they can ask questions to a tutor (a more experienced student teacher).
Also, learning resources like books or mathematical software, are chosen by the students. The software is given to the students as one possible tool. The included feedback should support the learning process in our setting. Only, if the included feedback is not sufficient, the students ask the tutor for further advise. This can either be done virtually via the integrated e-mail function or personally during the assisted tutorials.
Therefore, additional exploration exercises will be developed and the feedback capability of the laboratories will be extended. After a thoroughly evaluation of the tool in teaching, the concept will be carried to other sub-concepts of functions.

6. Acknowledgments

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